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Quantifying Stability and Chaoticity of One-dimensional and Two-dimensional Discrete Dynamical Systems using Stability Analysis and Lyapunov Exponents

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ABSTRACT

Discrete dynamical system is a system that evolve dynamically with discrete time. In this paper, we consider two discrete systems which exhibit chaotic behaviour. We show that the chaoticity of a system is depend on the values of parameter in the system. The objective of this paper is to investigate both stability and chaoticity of the systems using stability analysis and Lyapunov exponent, respectively. The results show that there is only one fixed point for the one-dimensional system, while for the two-dimensional system, there exist four possible fixed points. We have proved the stability conditions for each fixed point obtained. The Lyapunov results show that the one-dimensional system is stable when $r < \pi/2$ and chaotic when $r > \pi/2$. Whereas for two-dimensional systems considered, the system is chaotic for the whole range of parameter $\alpha \in (0,1]$. This study shows that it is significant to consider various values of parameter to study the stability of a dynamical systems in particular to control the chaos in a system.

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1. Introduction

Dynamical system can be characterized as a system that evolves with time and it comprises of potential states that decide the present state based on past state [1]. In fact, there are many applications of dynamical systems in real world such as the growth of crystals, stock market, electronic circuit, climate, brain, lasers, population dynamics and others that develop in time base on certain rules. The rules are deterministic as in learning of the state of a system at a specific moment of time which determines the condition of a system for every single future time.

The dynamical systems equations can be governed either by flows (with continuous time) or by the discrete maps (with discrete time). This paper focuses on the discrete maps which exhibit chaotic behaviour. The general form of discrete map is given by

$$x_{n+1} = f(x_n), \tag{1}$$

where $f: X \rightarrow X$ is a mapping from a topological space X to itself, $x_n \in X$ and $n \in \mathbb{Z}$ represents the time. Also, x_n means current state, while x_{n+1} means prediction for future state. An example for discrete map is the population function such as $f(x) = 2x$ which can be written in terms of a dynamical system that is $x_{n+1} = f(x_n) = 2x_n$, where x_n represents the numbers of the population at time n [2]. There are more studies which focus on discrete systems, for examples Roslan and Ashwin [3] investigated stability of attractor in a discrete piecewise linear expanding map while Mohd Roslan and Samsuddin [4] analyzed the stability of a discrete prey-predator system. Recently, Chen and Xie [5] analyze the stability of a two-dimensional discrete competitive system with exponential growth functions.

In this paper, we consider two examples of discrete systems which preserve chaotic behaviour. The main aim is to check the stability for both systems. The objectives are to investigate the stability condition as well as to establish chaotic behaviour existed in both maps using the method of Lyapunov exponent. This paper is organized as follows: in Section 2, some basic definitions in dynamical systems are introduced. Section 3 is devoted to discussing the two methods considered in this paper, they are: stability analysis and Lyapunov exponent. In Section 4, we prove the existence of fixed points as well as derive the stability conditions for fixed points and the formula of Lyapunov exponents. We then conclude our findings in the final section.

2. Research Background

In this section, we discuss some background concept on fixed points, orbits, periodic orbits as well as chaos.

2.1. Stability

Stability theory in mathematics means the stability of the solutions of differential equations and of directions of dynamical systems under little disturbances of initial states [6]. If a system is in a "stable state", it is called stationary system, whereas if a system is in "unstable state", it is called non-stationary system. When a system is in stationary states, it does not mean that there is no changes in the system after that. Consider the following definitions [7,8]:

Definition 1. A point $x_0 \in X$ is a *fixed point* for $f: X \rightarrow X$ if: $f(x_0) = x_0$. (2)

Definition 2. The *orbit* of a point $x_0 \in X$ is

$$\{f^n(x_0)\}_{n=0}^{\infty} = \{x_0, f(x_0), f(f(x_0)), \dots, f^n(x_0), \dots\}, \tag{3}$$

where f^n denotes the nth iterate of f, i.e.,

$$f^n(x_0) = f(f(\dots f(f(x_0)) \dots))), \; n \geq 0 \tag{4}$$

Definition 3. The *periodic orbit* of period n for f is defined as

$$f^n(x_0) = x_0, \; n \geq 0. \tag{5}$$

if $n = 1$, then we have the fixed point. When $n \geq 1$, we have period-n orbit.

Definition 4. A fixed point $x_0 \in X$ is *stable* if $|f'^{(x_0)}| < 1$ and *unstable* if $|f'^{(x_0)}| > 1$.

Definition 5. A fixed point $x_0 \in X$ is *hyperbolic* iff $|\lambda| \neq 1$ for all eigenvalues, λ, of the linearized system $Df(x_0)$.

2.2. Chaoticity

Chaos is a characteristic of dynamical system where it means that the system is totally unpredictable, has irregular pattern, uncertainty, random, complex and uncontrollable. According to Devaney [9], there are three main ingredients for a system to be chaotic:

i) The system f has sensitive dependence on initial conditions.
ii) f is topologically transitive which means it cannot be decomposed into two subsystems or invariant open subsets.
iii) There exist dense of periodic orbits in f.

According to Li and Yorke [10], for a system to have chaotic behavior, the system must have at least period three orbit. In this paper, we use the Lyapunov exponent to show the existence of chaotic behaviour in the system. If the largest value of Lyapunov exponent is positive, then the system is said to be chaotic, otherwise if the Lyapunov exponent is negative, then the system is non-chaotic, i.e. has stable behaviour.

3. Methodology

For the methodology, we first discuss stability condition of fixed point for discrete map. Then, we continue with the Lyapunov exponent.

3.1. Stability analysis for fixed points

To calculate the fixed points for map or discrete system, we let

$$f(x_*) = x_*. \tag{6}$$

For example,

$$\begin{aligned}
&f(x) = 2x(1-x),\\
&x_{n+1} = 2x_n(1-x_n),\\
&2x_*(1-x_*) = x_*,\\
&2x_* - 2{x_*}^2 = x_*,\\
&2{x_*}^2 - x_* = 0,\\
&x_*(2x_* - 1) = 0,\\
&x_{*1} = 0 \text{ or } x_{*2} = \tfrac{1}{2}.
\end{aligned} \tag{7}$$

Thus, the fixed points are 0 and $\frac{1}{2}$. Now, we proceed to find the stability of the two fixed points for maps by taking the derivatives for the original function $f(x_n)$.

$$\begin{aligned}
&f(x_n) = 2x_n(1-x_n),\\
&f(x_n) = 2x_n - 2{x_n}^2,\\
&f'(x_n) = 2 - 4x_n.
\end{aligned} \tag{8}$$

Then, substitute fixed points 0 and $\frac{1}{2}$ into $f'(x_n)$,

$$\begin{aligned}
&|f'(0)| = 2 - 4(0) = 2 \quad (> 1),\\
&\left|f'\left(\tfrac{1}{2}\right)\right| = 2 - 4\left(\tfrac{1}{2}\right) = 0 \quad (< 1).
\end{aligned} \tag{9}$$

It can be seen that the fixed point $x = 0$ is unstable since $|f'(0)| > 1$, while the fixed point $x = \frac{1}{2}$ is stable since $\left|f'\left(\frac{1}{2}\right)\right| < 1$. Here we also say that the point $x = \frac{1}{2}$ is an attractor while $x = 0$ is a repellor.

3.2. Lyapunov exponents

Lyapunov exponent (LE) is a classical method to prove whether a system is chaotic or not. Yet, this method proven to be useful and has been considered by many researchers until now. Suppose there are two nearby initial points in a space, x_0 and $x_0 + \Delta x_0$, each of the points which will create an orbit in that space utilizing some equation or system of equations. As time increases, i.e. as $n \to \infty$, we will investigate the long-time behaviour of the orbit for the points x_0 and $x_0 + \Delta x_0$. Figure 1 shows the sketch of for the Lyapunov exponent's concept [11].

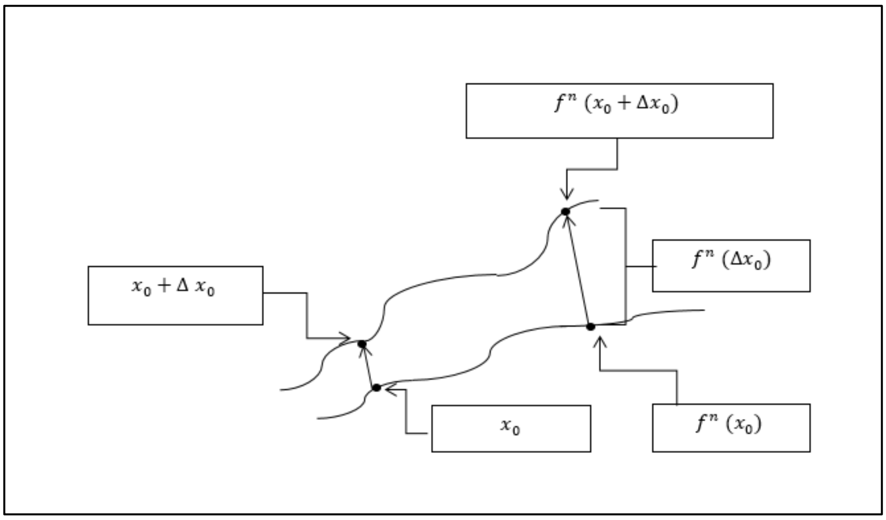

Figure 1. The schematic diagram for the concept of Lyapunov exponents

The formula of Lyapunov exponent is generally given as follows:

$$LE = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} ln \left| \frac{f^n(\Delta x_0)}{\Delta x_0} \right| \quad (10)$$

where f^n refers to nth iterates of f. The Lyapunov exponent is good for recognizing between the different types of orbits.

If $LE < 0$, it means that the orbit attracts to a stable fixed point or stable periodic orbit. This kind of systems show asymptotic stability. The more negative the exponent, the greater the stability. If $LE = 0$, it means shows that the system is in a type of steady state mode. If $LE > 0$, it means that the orbit is unstable and chaotic. No matter how proximate the nearby points are, it will eventually diverge to any arbitrary separation. All neighbourhoods in the phase space will at last be took in. These points are claimed to be unstable.

4. Numerical Examples and Results

In this section, we discuss two examples of discrete map, the one-dimensional and two-dimensional maps. We then apply the stability theory as well as the Lyapunov exponents for both maps.

4.1. One-dimensional discrete map

In this section, we consider a one-dimensional discrete map proposed by Dascalescu et al. [12]. The map is given by:

$$f_p(x) = \frac{2}{\pi} \arctan\big(\cot(\mathrm{rx})\big). \quad (11)$$

Here we show the calculation of fixed point, its stability and the Lyapunov exponent in more details. To find the fixed point, let the function, $f_p(x)$ in equation (1) equal to x.

$$f_p(x) = x$$
$$\frac{2}{\pi} tan^{-1}(\cot(rx)) = x$$
$$tan^{-1}(\cot(rx)) = \frac{\pi x}{2}$$
$$\tan\,(tan^{-1}(cot(rx))) = \tan\left(\frac{\pi x}{2}\right)$$
$$cot(rx) = \tan\left(\frac{\pi x}{2}\right)$$

$\frac{\cos(rx)}{\sin(rx)} = \frac{\sin\left(\frac{\pi x}{2}\right)}{\cos\left(\frac{\pi x}{2}\right)}$
$\cos(rx)\cos\left(\frac{\pi x}{2}\right) = \sin(rx)\sin\left(\frac{\pi x}{2}\right)$
$\cos(rx)\cos\left(\frac{\pi x}{2}\right) - \sin(rx)\sin\left(\frac{\pi x}{2}\right) = 0$ (12)

According to the trigonometric laws and identities,

$\cos(\mathrm{P}+\mathrm{Q}) = \cos(P)\cos(Q) - \sin(P)\sin(Q)$ (13)

By applying this law into the equation (2),
$\cos\left(rx + \frac{\pi x}{2}\right) = \cos(rx)\cos\left(\frac{\pi x}{2}\right) - \sin(rx)\sin\left(\frac{\pi x}{2}\right)$
$\cos\left(rx + \frac{\pi x}{2}\right) = 0$
$\left(rx + \frac{\pi x}{2}\right) = \cos^{-1} 0$
$\left(rx + \frac{\pi x}{2}\right) = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

For example, by taking $\left(rx + \frac{\pi x}{2}\right) = \frac{\pi}{2}$,

$rx + \frac{\pi x}{2} = \frac{\pi}{2}$
$2rx + \pi x = \pi$
$x(2r + \pi) = \pi$
$x_* = \frac{\pi}{2r+\pi}$. (14)

So the fixed point for f_p is $x_* = \frac{\pi}{2r+\pi}$. for example, if $r = 1$, then the fixed point of f_p is

$x_0 = \frac{\pi}{2+\pi}$. (15)

Then, proceed to find the differentiation of f_p

$f_p(x) = \frac{2}{\pi}\arctan(\cot(\mathrm{rx}))$
$f'_p(x) = \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{2}{\pi}\arctan(\cot(\mathrm{rx}))\right)$
$f'_p(x) = \frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{dx}}\left(\arctan(\cot(\mathrm{rx}))\right)$ (16)

Now, we apply the chain rule : $\frac{df(u)}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$
Let $f = \arctan(u)$
$u = \cot(rx)$
$\frac{\mathrm{d}}{\mathrm{du}}(\arctan \mathrm{u}) = \frac{1}{u^2+1}$
$\frac{\mathrm{d}}{\mathrm{dx}}(\cot(\mathrm{rx})) = -\mathrm{r}\csc^2(rx)$

Continue to differentiate $f_p(x)$;

$f'_p(x) = \frac{2}{\pi}\frac{d}{du}(\arctan u)\frac{d}{dx}(\cot(\mathrm{rx}))$

$$= \frac{2}{\pi}\frac{1}{u^2+1}\left(-\mathrm{rcsc}^2\,(rx)\right)$$
$$= \frac{2}{\pi}\frac{1}{\cot^2(\mathrm{rx})+1}\left(-\mathrm{rcsc}^2\,(rx)\right)$$
$$= -\frac{2\mathrm{rcsc}^2\,(rx)}{\pi(\cot^2(\mathrm{rx})+1)}$$
$$= \frac{-2\mathrm{r}}{\pi\sin^2 rx(\cot^2(\mathrm{rx})+1)} \quad (17)$$

By fixing that $r = 1$, and the fixed point $x_0 = \frac{\pi}{2+\pi}$,

$$\left|f'_p(x)\right| = \left|\frac{-2(1)}{\pi\sin^2\left(\frac{(1)\pi}{2+\pi}\right)\left(\cot^2\left(\frac{(1)\pi}{2+\pi}\right)+1\right)}\right|$$
$$\left|f'_p(x)\right| = 0.63662 \; < 1. \quad (18)$$

Since $\left|f'_p(t_k)\right| < 1$, it means that the fixed point is stable when $r = 1$. From the above equation, for any $r \in \left(0, \frac{\pi}{2}\right)$, all trajectories that begin nearby initial point x_0 converges accompanying to the attractor point x_0. However, for $r > \frac{\pi}{2}$, fixed point x_0 loses its stability which means it is unstable and chaos.

Theorem 1. The Lyapunov exponent for map $f_p(x)$ is positive when $r > \frac{\pi}{2}$ and negative when $r < \frac{\pi}{2}$.
Proof. We compute the Lyapunov exponent as follows:

$$LE = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} ln\left|f'_p(x_i)\right|$$
$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} ln\left|\frac{-2r}{\pi sin^2rx(1+cot^2rx)}\right|$$
$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} \ln\left(\frac{|-2r|}{\left|\pi sin^2rx\left(1+\frac{cos^2rx}{sin^2rx}\right)\right|}\right)$$
$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} \ln\left(\frac{2|r|}{\left|\pi sin^2rx\left(\frac{sin^2rx+cos^2rx}{sin^2rx}\right)\right|}\right)$$
$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} \ln\left(\frac{2|r|}{|\pi(1)|}\right)$$
$$= \ln\frac{2|r|}{\pi} \quad (19)$$

Since $|r| > 0$, then $\ln\frac{2|r|}{\pi} > 0$ if $r > \frac{\pi}{2}$ and $\ln\frac{2|r|}{\pi} < 0$ if $r < \frac{\pi}{2}$. This indicates that $f_p(x)$ is chaotic when $r > \frac{\pi}{2}$ and non-chaotic when $r < \frac{\pi}{2}$.

4.2. The two-dimensional competition species map

In this section, we consider a two-dimensional compeition system for which each species follows the logistic growth type. From Lopez-Ruiz and Fournier-Prunaret [13], the mentioned system is given by

$$\begin{aligned} x_{n+1} &= \alpha(-3y_n+4)x_n(1-x_n), \\ y_{n+1} &= \alpha(-3x_n+4)y_n(1-y_n), \end{aligned} \quad (20)$$

where α represent the strength of mutual competitive interaction. Suppose that the evolution of the above system can be written as follows: $f(x_n, y_n) = (x_{n+1}, y_{n+1})$.

The possible fixed points of this model are:

$$E_1 = (0,0),$$

$$E_2 = \left(0, \frac{4\alpha - 1}{4\alpha}\right),$$

$$E_3 = \left(\frac{4\alpha - 1}{4\alpha}, 0\right),$$

$$E_4 = \left(\frac{7\alpha + \sqrt{\alpha^2 + 12\alpha}}{6\alpha}, \frac{7\alpha + \sqrt{\alpha^2 + 12\alpha}}{6\alpha}\right),$$

$$E_5 = \left(\frac{7\alpha - \sqrt{\alpha^2 + 12\alpha}}{6\alpha}, \frac{7\alpha - \sqrt{\alpha^2 + 12\alpha}}{6\alpha}\right).$$

The fixed point E_1 means that both the competition species are extinct. Meanwhile, E_2 and E_3 shows the states where only one of the species will survive. Finally, E_4 and E_5 indicate where both species coexist. For every fixed point, we discuss the findings on the stability. The Jacobian matrix for system (20) is given by

$$Jf(x,y) = \begin{pmatrix} [D_x f_x](x,y) & [D_y f_x](x,y) \\ [D_x f_y](x,y) & [D_y f_y](x,y) \end{pmatrix} = \begin{pmatrix} \alpha(4-3y)(1-x) - \alpha(4-3y)x & -3\alpha x(1-x) \\ -3\alpha y(1-y) & \alpha(4-3x)y \end{pmatrix}.$$

Thus, the stability for all fixed points are stated in the following theorems:

Theorem 1. The fixed points $E_1, E_2. E_3$ are stable if and only if $\alpha < 1/4$ and unstable if $\alpha > 1/4$.
Proof. By substituting the fixed point E_1 into the Jacobian $Jf(x,y)$ and solve the determinant $Det(Jf - \lambda I) = 0$, the eigenvalues obtained are $\lambda_1 = \lambda_2 = 4\alpha$. For the fixed point to be stable, both eigenvalues must be less than 1, while it is unstable if at least one eigenvalue is greater than one. So E_1 is stable if and only if $|4\alpha| < 1$ which implies that $\alpha < 1/4$ and E_1 is unstable if and only if $|4\alpha| > 1$ which implies that $\alpha > 1/4$.

The eigenvalues for E_2 are: $\lambda_1 = -4\alpha + 2,\ \lambda_2 = \alpha + 3/4$. E_2 is stable if both eigenvalues less than 1, i.e. $= |-4\alpha + 2| < 1$ and $|\alpha + 3/4| < 1$. Solving these inequalities we obtain that E_2 is stable if $\alpha < 1/4$. It is then unstable if $\alpha > 1/4$. E_3 has the same eigenvalues as E_2, therefore the proof is the same.

Theorem 2. The coexistence fixed point E_4 is always unstable for all $\alpha \in (0,1]$.
Proof. The eigenvalues for E_4 are:

$$\lambda_{1,2} = \frac{\sqrt{\alpha^2+12\alpha}}{2} - \frac{\alpha}{2} + 2 \pm \frac{1}{3}\sqrt{A},$$

where $A = 8\alpha^2 + 60\,\alpha + 8\alpha\sqrt{\alpha^2 + 12\alpha} + 12\sqrt{\alpha^2 + 12\alpha} + 9$. For all values of $\alpha \in (0,1]$, one of the eigenvalues is greater than zero. This shows that E_4 is always unstable.

Theorem 3. The coexistence fixed point E_5 is stable if and only if $\alpha > 1/4$ and unstable if $\alpha < 1/4$.
Proof. The eigenvalues for E_5 are as follows:

$$\lambda_{1,2} = -\frac{\sqrt{\alpha^2+12\alpha}}{2} - \frac{\alpha}{2} + 2 \pm \frac{1}{3}\sqrt{B},$$

where $B = 8\alpha^2 + 60\,\alpha - 8\alpha\sqrt{\alpha^2 + 12\alpha} - 12\sqrt{\alpha^2 + 12\alpha} + 9$. When $\alpha < 1/4$, both $\lambda_{1,2} > 1$. On the other hand, when $\alpha > 1/4$, both $\lambda_{1,2} < 1$. This means that the coexistence fixed point E_5 is unstable when α is small and stable when α is large.

Now, we compute the Lyapunov exponent. Since the map (20) is a two-dimensional system, we therefore employ the concept of tangential and normal Lyapunov exponents, where this means that we compute the Lyapunov exponents in x-direction and y-direction respectively. We follow this approach from [14,15,16].

Theorem 4. Suppose t is any positive number. For all values of $\alpha \in (0,1]$, the tangential Lyapunov exponent is positive.
Proof. To compute the Lyapunov exponent in x-direction, we take the first component of the Jacobian $Jf(x,y)$ and take $y = 0$. Thus,

$$\begin{aligned}\lambda_{||} &= \int_0^t \ln|[D_x f_x](x,0)|\,dx \\ &= \int_0^t \ln|4\alpha(1-x) - 4\alpha x|\,dx \\ &= -4\alpha t^2 + 4\alpha t \\ &= 4\alpha t(1-t) > 0\end{aligned}$$

for all $\alpha \in (0,1]$.

Theorem 5. For all values of $\alpha \in (0,1]$, the normal Lyapunov exponent is positive.
Proof. To compute the Lyapunov exponent in y-direction, we take the fourth component of the Jacobian $Jf(x,y)$ and take $y = 0$. Thus,

$$\begin{aligned}\lambda_{\perp} &= \int_0^1 \ln\left|\left[D_y f_y\right](x,0)\right|\,dx \\ &= \int_0^1 \ln|\alpha(4-3x)|\,dx \\ &= \frac{5}{2}\alpha > 0\end{aligned}$$

for all $\alpha \in (0,1]$.

Since both $\lambda_{||}$ and $\lambda_{\perp}$are positive, this shows that system (20) is chaotic.

5. Conclusion

For the one-dimensional map, the work in this paper was motivated by Dascalescu et al. [12] where we show the calculation more details. First, the calculation for the fixed point was carried out. We then use the stability theory to determine the stability condition for this point. We observed that the fixed point x_* is stable when $r < \frac{\pi}{2}$. However, for $r > \frac{\pi}{2}$, the fixed point now is unstable. Secondly, by conducting Lyapunov exponent for the discrete map, we found out that the Lyapunov exponent is negative when the parameter $r < \frac{\pi}{2}$. This means that the system is said to be stable when the parameter r within the interval $\left(0, \frac{\pi}{2}\right)$. On the other hand, when r passes $\frac{\pi}{2}$, the Lyapunov exponent becomes positive. Positivity of Lyapunov exponent means that the system is now chaos.

Furthermore, the second model is motivated from the work by Lopez-Ruiz and Fournier-Prunaret [13] where their model is a biological model of competition species. We have analyzed the stability of all fixed points for this map. Interesting results have been observed in a way that when the mutual competitive strength $\alpha < 1/4$, the fixed points of extinction of both species or one of the species survive (mutual exclusion) are stable. This means that for low α , these fixed points might occur in the future. In contrast, both species exist (E_5) is unstable under the same condition, but stable when $\alpha > 1/4$. Biologically interpret, this means that both species that compete each other will survive if the level of competition strength is high. Meanwhile, another coesitence fixed point E_4 is always unstable for $\alpha \in (0,1]$. Within this range, the Lyapunov exponent results have shown that this competition system is also chaotic.

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